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Nonparametric Estimation of Location Parameter after a Preliminary Test on Regression in the Multivariate Case*

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For a simple multivariate regression model, nonparametric estimation of the (vector of) intercept following a preliminary test on the regression vector is considered. Along with the asymptotic distribution of these estimators, their asymptotic bias and dispersion matrices are studied and allied efficiency results are presented.

1. INTRODUCTION

Let $\{\mathbf{X}_i = (X_{i1}, \ldots, X_{ip})', i \geqslant 1\}$ be a sequence of independent random vectors (rv) with continuous p ($\geqslant 1$)-variate distribution functions (df) $\{F_i, i \geqslant 1\}$ where, for every $i \geqslant 1$,

$$F_i(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\theta} - \boldsymbol{\beta} t_i), \quad \mathbf{x} \in E^p, \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)', \quad \boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)', \tag{1.1}$$

F, $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ are unknown and $\{t_i, i \geqslant 1\}$ is a sequence of known constants. We are primarily interested in the estimation of the location parameter $\boldsymbol{\theta}$. A general class of rank order estimators of $\boldsymbol{\theta}$ (when $\boldsymbol{\beta}$ may or may not be known) has been studied by Sen and Puri [5]. The estimator of $\boldsymbol{\theta}$ is different in the

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two situations where $\boldsymbol{\beta}$ is specified or not. Let $\hat{\boldsymbol{\theta}}_n$ and $\tilde{\boldsymbol{\theta}}_n$ be respectively the estimator of $\boldsymbol{\theta}$, based on $X_1, \ldots, X_n$, when we assume that $\boldsymbol{\beta}$ is equal to $\mathbf{0}$ and when $\boldsymbol{\beta}$ is not specified. Usually, $\tilde{\boldsymbol{\theta}}_n$ has a larger dispersion than $\hat{\boldsymbol{\theta}}_n$ when $\boldsymbol{\beta} = \mathbf{0}$, while, for $\boldsymbol{\beta} \neq \mathbf{0}$, $\hat{\boldsymbol{\theta}}_n$ may not be a desirable estimator of $\boldsymbol{\theta}$. When $\boldsymbol{\beta}$ is unknown but is suspected to be close to $\mathbf{0}$, often, a preliminary test on $\boldsymbol{\beta}$ is made: if H_0: $\boldsymbol{\beta} = \mathbf{0}$ is tenable, then $\hat{\boldsymbol{\theta}}_n$ is used, while, if H_0 is rejected, $\tilde{\boldsymbol{\theta}}_n$ is used. We denote such a preliminary test estimator (PTE) by $\boldsymbol{\theta}_n^*$. The object of the present investigation is to study the asymptotic bias and dispersion matrix of these three estimators when $\boldsymbol{\beta}$ is close to $\mathbf{0}$. In the univariate case, Saleh and Sen [4] have considered this problem and the present paper extends their results to the multivariate case. In view of the methodological similarity, in many places, we shall omit some details of derivations by cross references to Saleh and Sen [4] and concentrate mainly on the new results.

Along with the preliminary notions and basic assumptions, the estimators are formally introduced in Section 2. Section 3 deals with their asymptotic distributions. Expressions for the asymptotic bias and dispersion matrix of these estimators are obtained in Section 4 and a comparative study of these quantities is made in the last section.

2. Preliminary Notions and the Estimators

Let $R_{ni}^{(j)}(a, b)$ (or $R_{ni}^{(j)*}(a, b)$) be the rank of $X_{ij} - a - bt_i$ (or $|X_{ij} - a - bt_i|$) among $X_{1j} - a - bt_1, \ldots, X_{nj} - a - bt_n$ (or $|X_{1j} - a - bt_1|, \ldots, |X_{nj} - a - bt_n|$), for $i = 1, \ldots, n$, $j = 1, \ldots, p$, where a and b are real numbers. Let then $\mathbf{T}_n(\mathbf{a}, \mathbf{b}) = (T_{n,1}(a_1, b_1), \ldots, T_{n,p}(a_p, b_p))'$ and $S_n(\mathbf{a}, \mathbf{b}) = (S_{n,1}(a_1, b_1), \ldots, S_{n,p}(a_p, b_p))'$, where, for each j $(= 1, \ldots, p)$,

$$T_{n,j}(a, b) = n^{-1} \sum_{i=1}^{n} \operatorname{sgn}(X_{ij} - a - bt_i)\, a_{n,j}^*(R_{ni}^{(j)*}(a, b)), \tag{2.1}$$

$$S_{n,j}(a, b) = n^{-1} \sum_{i=1}^{n} (t_i - \bar{t}_n)\, a_{n,j}(R_{ni}^{(j)}(a, b)), \tag{2.2}$$

$\bar{t}_n = n^{-1} \sum_{i=1}^{n} t_i$ and the scores $a_{n,j}(i)$ and $a_{n,j}^*(i)$ are defined by

$$\begin{aligned} a_{n,j}(i) &= E\phi_j(U_{ni}) \quad \text{or} \quad \phi_j(i/(n+1)) \\ \text{and} \qquad & \\ a_{n,j}^*(i) &= E\phi_j^*(U_{ni}) \quad \text{or} \quad \phi_j^*(i/(n+1)), \end{aligned} \tag{2.3}$$

for $i = 1, \ldots, n$, where $U_{n1} < \cdots < U_{nn}$ are the ordered rv's of a sample of size n from the uniform (0, 1) df and the score function $\phi_j = \{\phi_j(u), u \in (0, 1)\}$ is absolutely continuous, non-decreasing, skew-symmetric (i.e., $\phi_j(u) +$

$\phi_j(1-u) = 0$, $\forall u \in (0, 1))$ and square integrable inside (0, 1), while $\phi_j^*(u) = \phi_j((1+u)/2)$, $u \in (0, 1)$, $j = 1,..., p$.

We assume that the t_i are all bounded and define $Q_n = \sum_{i=1}^n (t_i - \bar{t}_n)^2$ and $Q_n^* = Q_n/n$. Further, we assume that

$$\lim_{n\to\infty} \bar{t}_n = \bar{t}(|\bar{t}| < \infty) \quad \text{and} \quad \lim_{n\to\infty} Q_n^* = Q^*(0 < Q^* < \infty) \quad \text{both exist.} \tag{2.3}$$

Finally, let $\mathscr{F}_p$ be the class of all p-variate absolutely continuous df's which are diagonally symmetric about 0 and have a finite Fisher information matrix. Then, we assume that F in (1.1) belongs to $\mathscr{F}_p$.

Note that $T_{n,j}(a, b)$ is $\searrow$ in a (for a given b) and $S_{n,j}(a, b)$ is $\searrow$ in b (independently of a) for, every $j = 1,..., p$. Also, under H_0^*: $\boldsymbol{\theta} = \boldsymbol{\beta} = \mathbf{0}$ $\mathbf{T}_n(\mathbf{0}\ \mathbf{0})$ and $\mathbf{S}_n(\mathbf{0}, \mathbf{0})$ both have mean $\mathbf{0}$. As such, as in Sen and Puri [5], we let

$$\hat{\theta}_{n,j}^{(1)} = \sup\{a: T_{n,j}(a, 0) > 0\}, \qquad \hat{\theta}_{n,j}^{(2)} = \inf\{a: T_{n,j}(a, 0) < 0\}; \tag{2.4}$$

$$\hat{\theta}_{n,j} = \tfrac{1}{2}(\hat{\theta}_{n,j}^{(1)} + \hat{\theta}_{n,j}^{(2)}), \quad j = 1,..., p; \qquad \hat{\boldsymbol{\theta}}_n = (\hat{\theta}_{n,1}, ..., \hat{\theta}_{n,p})'; \tag{2.5}$$

$$\hat{\beta}_{n,j}^{(1)} = \sup\{b: S_{n,j}(0, b) > 0\}, \qquad \hat{\beta}_{n,j}^{(2)} = \inf\{b: S_{n,j}(0, b) < 0\}; \tag{2.6}$$

$$\hat{\beta}_{n,j} = \tfrac{1}{2}(\hat{\beta}_{n,j}^{(1)} + \hat{\beta}_{n,j}^{(2)}), \quad j = 1,..., p; \qquad \hat{\boldsymbol{\beta}}_n = (\hat{\beta}_{n,1}, ..., \hat{\beta}_{n,p})'; \tag{2.7}$$

$$\tilde{\theta}_{n,j}^{(1)} = \sup\{a: T_{n,j}(a, \hat{\beta}_{n,j}) > 0\}, \qquad \tilde{\theta}_{n,j}^{(2)} = \inf\{a: T_{n,j}(a, \hat{\beta}_{n,j}) < 0\}; \tag{2.8}$$

$$\tilde{\theta}_{n,j} = \tfrac{1}{2}(\tilde{\theta}_{n,j}^{(1)} + \tilde{\theta}_{n,j}^{(2)}), \quad j = 1,..., p; \qquad \tilde{\boldsymbol{\theta}}_n = (\tilde{\theta}_{n,1}, ..., \tilde{\theta}_{n,p})'. \tag{2.9}$$

Then, $\hat{\boldsymbol{\theta}}_n$ is a translation-invariant, robust and consistent estimator of $\boldsymbol{\theta}$ when $\boldsymbol{\beta} = \mathbf{0}$ and $\tilde{\boldsymbol{\theta}}_n$ is a similar estimator of $\boldsymbol{\theta}$ when $\boldsymbol{\beta}$ is unspecified. To formulate the PTE $\boldsymbol{\theta}_n^*$, we let $R_{ni}^{(j)} = R_{ni}^{(j)}(0, 0)$, $j = 1,..., p$, $i = 1,..., n$ and define $\mathbf{M} = ((m_{j\ell}))_{j,\ell=1,...,p}$ by letting

$$m_{j\ell} = n^{-1} \sum_{i=1}^{n} [a_{n,j}(R_{ni}^{(j)}) - \bar{a}_{n,j}][a_{n,\ell}(R_{ni}^{(\ell)}) - \bar{a}_{n,\ell}], \qquad j, \ell = 1,..., p, \tag{2.10}$$

where $\bar{a}_{n,j} = n^{-1} \sum_{i=1}^n a_{n,j}(i)$, $j = 1,..., p$. As in Puri and Sen [2], for testing H_0: $\boldsymbol{\beta} = \mathbf{0}$, we consider the test statistic

$$\mathscr{L}_n = (n\mathbf{S}_n'\mathbf{M}_n^{-}\mathbf{S}_n)/Q_n^* \qquad \text{where} \quad \mathbf{S}_n = \mathbf{S}_n(0, 0) \tag{2.11}$$

and $\mathbf{M}_n^-$ is the generalized inverse of $\mathbf{M}$; $\mathscr{L}_n$ is a conditionally distribution-free statistic and for large n, under H_0, it has closely the chi-square distribution with p degrees of freedom when $\mathbf{M}_n$ is of full rank. We denote the upper $100\alpha\%$ point of the null distribution of $\mathscr{L}_n$ by $\mathscr{L}_{n,\alpha}$ and note that $\mathscr{L}_{n,\alpha} \to \chi_{p,\alpha}^2$, the upper $100\alpha\%$ point of the central chi-square df with p degrees of freedom. Then, our proposed PTE is defined by

$$\boldsymbol{\theta}_n^* = \begin{cases} \hat{\boldsymbol{\theta}}_n, & \text{if } \mathscr{L}_n \leqslant \mathscr{L}_{n,\alpha} \\ \tilde{\boldsymbol{\theta}}_n, & \text{if } \mathscr{L}_n > \mathscr{L}_{n,\alpha}. \end{cases} \tag{2.12}$$

As is usually the case with PTE, $\boldsymbol{\theta}_n^*$ is generally neither an unbiased estimator of $\boldsymbol{\theta}$ nor it is asymptotically multinormal (though $\hat{\boldsymbol{\theta}}_n$ and $\tilde{\boldsymbol{\theta}}_n$ are so). For this reason, we intend to study the asymptotic properties of all these estimators.

3. Asymptotic Distribution Theory of the Estimators

Since the preliminary test estimator is of interest when $\boldsymbol{\beta}$ is suspected to be close to $\mathbf{0}$, we confine ourselves to local alternatives $\{K_n\}$, where

$$K_n\colon \boldsymbol{\beta} = \boldsymbol{\beta}_{(n)} = n^{-1/2}\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_p)'(\text{fixed}) \in E^p. \tag{3.1}$$

Also, let $F_{[j]}$ and $F_{[j\ell]}$ be respectively the marginal df of the jth variate and the joint df of the (j, ℓ)th variates [corresponding to the df F in (1.1)] and let

$$\lambda_{j\ell} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_j(F_{[j]}(x))\,\phi_\ell(F_{[\ell]}(y))\,dF_{[j\ell]}(x, y) \qquad \text{for } j, \ell = 1, \ldots, p; \quad (3.2$$

$$\boldsymbol{\Lambda} = ((\lambda_{j\ell})), \qquad \tau_{j\ell} = \lambda_{j\ell}/(\mu_j^*\mu_\ell^*) \qquad \text{for } j, \ell = 1, \ldots, p; \qquad \mathbf{T} = ((\tau_{j\ell},) \quad (3.2)$$

where

$$\mu_j^* = \int_0^1 \phi_j(u)\,\psi_j(u)\,du \qquad \text{and} \qquad \psi_j(u) = -f_{[j]}'(F_{[j]}^{-1}(u))/f_{[j]}(F_{[j]}^{-1}(u)),$$
$$u \in (0, 1), \quad (3.3)$$

for $j = 1, \ldots, p$. We assume that $\boldsymbol{\Lambda}$ and $\mathbf{T}$ are both of full rank. Then, the following theorems hold.

Theorm 3.1. *Under $\{K_n\}$ in* (3.1) *and the assumed regularity conditions, as $n \to \infty$,*

$$n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \xrightarrow[\mathscr{D}]{} \mathscr{N}_p(\bar{t}\boldsymbol{\gamma}, \mathbf{T}) \tag{3.4}$$

and

$$n^{1/2}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \xrightarrow[\mathscr{D}]{} \mathscr{N}_p(\mathbf{0}, (1 + \bar{t}^2/Q^*)\mathbf{T}). \tag{3.5}$$

THEOREM 3.2. *Under $\{K_n\}$ in* (3.1) *and the regularity conditions of Theorem* 3.1, *as* $n \to \infty$,

$$G_p^*(\mathbf{x}; \boldsymbol{\gamma}) = \lim_{n\to\infty} P_{K_n}\{n^{1/2}(\boldsymbol{\theta}_n^* - \boldsymbol{\theta}) \leqslant \mathbf{x}\} = G_p(\mathbf{x} - \bar{t}\boldsymbol{\gamma}; \mathbf{0}, \mathbf{T})\, H_p(\chi^2_{p,\alpha}; \Delta^*)$$

$$+ \int_{E(\boldsymbol{\gamma})} G_p(\mathbf{x} + (\bar{t}/(Q^*)^{1/2})\mathbf{z}; \mathbf{0}, \mathbf{T})\, dG_p(\mathbf{z}; \mathbf{0}, \mathbf{T}), \qquad \forall \mathbf{x} \in E^p, \quad (3.6)$$

where $G_p(\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ *is the p-variate normal* df *with mean vector* $\boldsymbol{\mu}$ *and dispersion matrix* $\boldsymbol{\Sigma}$, $H_p(x; \delta)$ *is the non-central chi square* df *with p degrees of freedom and non-centrality parameter* δ, $E(\boldsymbol{\gamma}) = \{\mathbf{y} \in E^p: (\mathbf{y} + (Q^*)^{1/2}\boldsymbol{\gamma})' \,\mathbf{T}^{-1}(\mathbf{y} + (Q^*)^{1/2}\boldsymbol{\gamma}) > \chi^2_{p,\alpha}\}$ *and* $\Delta^* = Q^*(\boldsymbol{\gamma}'\mathbf{T}^{-1}\boldsymbol{\gamma})$.

The proof of Theorem 3.1 follows directly from Sen and Puri [5] while Theorem 3.2 is a direct multivariate generalization of Theorem 3.2 of Saleh and Sen [4] and, in view of the similarity, the details of the proof are omitted.

For latter use, we denote the probability density function (pdf) corresponding to G_p^* by g_p^*, so that for every $\mathbf{x} \in E^p$,

$$g_p^*(\mathbf{x}; \boldsymbol{\gamma}) = g_p(\mathbf{x} - \bar{t}\boldsymbol{\gamma}; \mathbf{0}, \mathbf{T})\, H_p(\chi^2_{p,\alpha}; \Delta^*)$$

$$+ \int_{E(\boldsymbol{\gamma})} g_p(\mathbf{x} + (\bar{t}/(Q^*)^{1/2})\mathbf{z}; \mathbf{0}, \mathbf{T})\, dG_p(\mathbf{z}; \mathbf{0}, \mathbf{T}) \qquad (3.7)$$

where g_p stands for the multinormal pdf.

4. ASYMPTOTIC BIAS AND DISPERSION MATRIX OF THE ESTIMATORS

The mean vector and the dispersion matrix of the asymptotic df in (3.4), (3.5) and (3.6) are defined as the *asymptotic bias* and *asymptotic dispersion matrix* (a.d.m.) of the estimators $\hat{\boldsymbol{\theta}}_n$, $\tilde{\boldsymbol{\theta}}_n$ and $\boldsymbol{\theta}_n^*$, respectively. Thus, we have

$$\begin{aligned} \boldsymbol{\xi}_1(\boldsymbol{\gamma}) &= \text{asymptotic bias of } n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) = \bar{t}\boldsymbol{\gamma}, \\ \boldsymbol{\xi}_2(\boldsymbol{\gamma}) &= \text{asymptotic bias of } n^{1/2}(\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) = \mathbf{0}, \qquad \forall \boldsymbol{\gamma}; \end{aligned} \qquad (4.1)$$

$$\boldsymbol{\Gamma}_1(\boldsymbol{\gamma}) = \text{a.d.m. of } n^{1/2}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) = \mathbf{T} + \bar{t}^2\boldsymbol{\gamma}\boldsymbol{\gamma}',$$

and hence, by differentiating with respect to **a**, it follows by some simple steps that

$$\int_{(\mathbf{x}+\mathbf{a})'\mathbf{B}^{-1}(\mathbf{x}+\mathbf{a})>c} \mathbf{x}\, dG_p(\mathbf{x}; \mathbf{0}, \mathbf{B}) = \mathbf{a}[H_p(c; \delta) - H_{p+2}(c; \delta)], \tag{4.4}$$

$$\begin{aligned}\int_{\mathbf{x}'\mathbf{B}^{-1}\mathbf{x}>c} &(\mathbf{x}-\mathbf{a})(\mathbf{x}-\mathbf{a})'\, dG_p(\mathbf{x}; \mathbf{a}, \mathbf{B}) \\ &= \int_{(\mathbf{x}+\mathbf{a})'\mathbf{B}^{-1}(\mathbf{x}+\mathbf{a})>c} \mathbf{x}\mathbf{x}'\, dG_p(\mathbf{x}; \mathbf{0}, \mathbf{B}) \\ &= [1 - H_{p+2}(c; \delta)]\mathbf{B} - \mathbf{a}\mathbf{a}'\{H_p(c; \delta) - 2H_{p+2}(c; \delta) + H_{p+4}(c; \delta)\}]. \end{aligned}\tag{4.5}$$

Hence,

$$\begin{aligned}\boldsymbol{\xi}^*(\boldsymbol{\gamma}) &= \text{asymptotic bias of } n^{1/2}(\boldsymbol{\theta}_n^* - \boldsymbol{\theta}) \\ &= \bar{t}\boldsymbol{\gamma} H_p(\chi^2_{p,\alpha}; \Delta^*) + \int_{E(\boldsymbol{\gamma})} (-\bar{t}/(Q^*)^{1/2})\mathbf{z}\, dG_p(\mathbf{z}; \mathbf{0}, \mathbf{T}) \\ &= \bar{t}\boldsymbol{\gamma} H_p(\chi^2_{p,\alpha}; \Delta^*) \\ &\quad + (-\bar{t}/(Q^*)^{1/2})(Q^*)^{1/2}\, \boldsymbol{\gamma}[H_p(\chi^2_{p,\alpha}; \Delta^*) - H_{p+2}(\chi^2_{p,\alpha}; \Delta^*)] \\ &= \{\bar{t} H_{p+2}(\chi^2_{p,\alpha}; \Delta^*)\}\boldsymbol{\gamma}, \end{aligned}\tag{4.6}$$

$$\begin{aligned}\boldsymbol{\Gamma}^*(\boldsymbol{\gamma}) &= \text{a.d.m. of } n^{1/2}(\boldsymbol{\theta}_n^* - \boldsymbol{\theta}) \\ &= (1 + \bar{t}^2/Q^*)\mathbf{T} - (\bar{t}^2/Q^*)\, H_{p+2}(\chi^2_{p,\alpha}; \Delta^*)\mathbf{T} \\ &\quad - \bar{t}^2[H_p(\chi^2_{p,\alpha}; \Delta^*) - 2H_{p+2}(\chi^2_{p,\alpha}; \Delta^*) + H_{p+4}(\chi^2_{p,\alpha}; \Delta^*)]\, \boldsymbol{\gamma}\boldsymbol{\gamma}'. \end{aligned}\tag{4.7}$$

We conclude this section with the remark that **T**, appearing in (4.2) is positive definite, while, $\boldsymbol{\gamma}\boldsymbol{\gamma}'$ in (4.7) is positive semi-definite (of rank 1), so that if $\ell_1 \geqslant \cdots \geqslant \ell_p$ be the characteristic roots of $\mathbf{T}^{-1}\boldsymbol{\gamma}\boldsymbol{\gamma}'$, then $\ell_2 = \cdots = \ell_p = 0$ and

$$\ell_1 = \text{Trace}[\mathbf{T}^{-1}\boldsymbol{\gamma}\boldsymbol{\gamma}'] = \boldsymbol{\gamma}'\mathbf{T}^{-1}\boldsymbol{\gamma} = \Delta^*/Q^*. \tag{4.8}$$

5. Comparison of the Estimators

It follows from (4.1), (4.2), (4.6) and (4.7) that for $\bar{t} = 0$,

$$\boldsymbol{\xi}_1(\boldsymbol{\gamma}) = \boldsymbol{\xi}_2(\boldsymbol{\gamma}) = \boldsymbol{\xi}^*(\boldsymbol{\gamma}) = \mathbf{0} \quad \text{and} \quad \boldsymbol{\Gamma}_1(\boldsymbol{\gamma}) = \boldsymbol{\Gamma}_2(\boldsymbol{\gamma}) = \boldsymbol{\Gamma}^*(\boldsymbol{\gamma}) = \mathbf{T}, \quad \forall \boldsymbol{\gamma}. \tag{5.1}$$

Also, for $\bar{t} = 0$, by (3.6), $G_p^*(x; \boldsymbol{\gamma}) = G_p(x; \mathbf{0}, \mathbf{T})$, $\forall \boldsymbol{\gamma}$, so that all the three estimators have the same asymptotic (multinormal) df; their common asymptotic bias is equal to $\mathbf{0}$ and common a.d.m. is equal to $\mathbf{T}$. On the other hand, for $\bar{t} \neq 0$, $\xi_2(\gamma) = \mathbf{0}$ and $\mathbf{\Gamma}_2(\gamma)$ does not depend on γ while the other quantities do so.

THEOREM 5.1. *Under H_0: $\beta = \mathbf{0}$ (i.e., $\gamma = \mathbf{0}$), when $\bar{t} \neq 0$, $\mathbf{\Gamma}_2(\mathbf{0}) - \mathbf{\Gamma}_1(\mathbf{0})$, $\mathbf{\Gamma}_2(\mathbf{0}) - \mathbf{\Gamma}^*(\mathbf{0})$ and $\mathbf{\Gamma}^*(\mathbf{0}) - \mathbf{\Gamma}_1(\mathbf{0})$ are all positive definite whenever $\mathbf{T}$ is so.*

Proof. By (4.2), for $\gamma = \mathbf{0}$, $\mathbf{\Gamma}_2(\mathbf{0}) - \mathbf{\Gamma}_1(\mathbf{0}) = (\bar{t}^2/Q^*)\mathbf{T}$ and by (4.2) and (4.7), $\mathbf{\Gamma}_2(\mathbf{0}) - \mathbf{\Gamma}^*(\mathbf{0}) = (\bar{t}^2/Q^*)\, H_{p+2}(\chi^2_{p,\alpha}\,; 0)\mathbf{T}$ and $\mathbf{\Gamma}^*(\mathbf{0}) - \mathbf{\Gamma}_1(\mathbf{0}) = (\bar{t}^2/Q^*)\{1 - H_{p+2}(\chi^2_{p,\alpha}\,; 0)\}\mathbf{T}$ where $\bar{t}^2/Q^* > 0$ and $0 < H_{p+2}(\chi^2_{p,\alpha}\,; 0) < H_p(\chi^2_{p,\alpha}\,; 0) = 1 - \alpha < 1$. Hence, the result follows.

We define the asymptotic relative efficiency (A.R.E.) in terms of the *asymptotic generalized variance* (i.e., the pth root of the determinant of the a.d.m.) [as in Chapter 6 of Puri and Sen [3]]. Then, for $\gamma = \mathbf{0}$, the A.R.E. of $\{\theta_n^*\}$ with respect to $\{\hat{\theta}_n\}$ is given by

$$e(\theta^*, \hat{\theta} \mid \gamma = \mathbf{0}) = \{|\,\mathbf{\Gamma}_1(\mathbf{0})|/|\,\mathbf{\Gamma}^*(\mathbf{0})|\}^{1/p} = \{1 + (\bar{t}^2/Q^*)[1 - H_{p+2}(\chi^2_{p,\alpha}\,; 0)]\}^{-1}, \tag{5.2}$$

where $|\,\mathbf{B}\,|$ stands for the determinant of $\mathbf{B}$ (= product of its characteristic roots). Similarly,

$$\begin{aligned} e(\theta^*, \tilde{\theta} \mid \gamma = 0) &= \{|\,\mathbf{\Gamma}_2(\mathbf{0})|/|\,\mathbf{\Gamma}^*(\mathbf{0})|\}^{1/p} \\ &= (1 + \bar{t}^2/Q^*)\{1 + (\bar{t}^2/Q^*)[1 - H_{p+2}(\chi^2_{p,\alpha}\,; 0)]\}^{-1}. \end{aligned} \tag{5.3}$$

Thus,

$$e(\theta^*, \hat{\theta} \mid \gamma = \mathbf{0}) \leqslant 1 \leqslant e(\theta^*, \tilde{\theta} \mid \gamma = \mathbf{0}), \qquad \text{for every } \bar{t}, \tag{5.4}$$

where, for $\bar{t} = 0$, both the inequality signs reduce to = signs.

Let us next consider the case of $\gamma \neq \mathbf{0}$. Form (4.2), (4.7) and (4.8), we have

$$\begin{aligned} e(\hat{\theta}, \tilde{\theta} \mid \gamma) &= \{|\,\mathbf{\Gamma}_2(\gamma)|/|\,\mathbf{\Gamma}_1(\gamma)|\}^{1/p} = (1 + \bar{t}^2/Q^*)\{|\,\epsilon + \bar{t}^2\mathbf{T}^{-1}\gamma\gamma'\,|\}^{-1/p} \\ &= (1 + \bar{t}^2/Q^*)\left\{\prod_{j=1}^{p} (1 + \bar{t}^2\ell_j)\right\}^{-1/p} = (1 + \bar{t}^2/Q^*)(1 + \bar{t}^2\ell_1)^{-1/p} \\ &= (1 + \bar{t}^2/Q^*)(1 + \bar{t}^2\Delta^*/Q^*)^{-1/p}. \end{aligned} \tag{5.5}$$

Thus, (5.5) assumes constant values on the ellipsoids specified by $\Delta^* = \gamma'\mathbf{T}^{-1}\gamma$ and as $\Delta^* \to 0$, it goes to $(1 + \bar{t}^2/Q^*)$, while it goes to 0 as $\Delta^* \to \infty$, indicating that for large Δ^*, $\hat{\theta}_n$ becomes relatively inefficient (mainly because its asymptotic bias $\gamma\bar{t}$ shoots up its a.d.m.). In fact,

$$e(\hat{\theta}, \tilde{\theta} \mid \gamma) \text{ is } \gtreqless 1 \text{ according as } \Delta^* \text{ is } \gtreqless [(1 + \bar{t}^2/Q^*)^p - 1]/(\bar{t}^2/Q^*). \tag{5.6}$$

Let us now define, for every $\delta \geqslant 0$ and $0 < \alpha < 1$,

$$\rho_{1,p}(\delta, \alpha) = 1 - H_{p+2}(\chi^2_{p,\alpha}\,; \delta)$$
and (5.7)
$$\rho_{2,p}(\delta, \alpha) = H_p(\chi^2_{p,\alpha}\,; \delta) - 2H_{p+2}(\chi^2_{p,\alpha}\,; \delta) + H_{p+4}(\chi^2_{p,\alpha}\,; \delta).$$

Then, from (4.2), (4.7) and (5.7), we have

$$\begin{aligned} e(\boldsymbol{\theta}^*, \hat{\boldsymbol{\theta}} \mid \boldsymbol{\gamma}) &= \{|\boldsymbol{\Gamma}_1(\boldsymbol{\gamma})|/|\boldsymbol{\Gamma}^*(\boldsymbol{\gamma})|\}^{1/p} \qquad (5.8)\\ &= \{|\mathbf{I} + \bar{t}^2\mathbf{T}^{-1}\boldsymbol{\gamma}\boldsymbol{\gamma}'|\}^{1/p}\{|\mathbf{I}(1 + \rho_{1,p}(\Delta^*, \alpha)\,\bar{t}^2/Q^*) \\ &\qquad - \rho_{2,p}(\Delta^*, \alpha)\,\bar{t}^2\mathbf{T}^{-1}\boldsymbol{\gamma}\boldsymbol{\gamma}'|\}^{-1/p} \\ &= [(1 + \Delta^*\bar{t}^2/Q^*)/(1 + (\bar{t}^2/Q^*)\{\rho_{1,p}(\Delta^*, \alpha) \\ &\qquad - \rho_{2,p}(\Delta^*, \alpha)\,\Delta^*\})(1 + \rho_{1,p}(\Delta^*, \alpha)\,\bar{t}^2/Q^*)^{p-1}]^{1/p}. \end{aligned}$$

(5.8) also assumes constant values on the ellipsoids specified by $\Delta^* = \boldsymbol{\gamma}'\mathbf{T}^{-1}\boldsymbol{\gamma}$. Note that $0 < \alpha < \rho_{1,p}(\Delta^*, \alpha) \leqslant 1$ for every $\Delta^* \geqslant 0$ and it converges to 1 as $\Delta^* \to \infty$, while

$$\rho_{2,p}(\Delta^*, \alpha) = (\exp\{-\tfrac{1}{2}\Delta^*\})\left[\sum_{r=0}^{\infty} (\tfrac{1}{2}\Delta^*)^r (r!)^{-1}\, \rho^{(p)}_{2,p+2r}(0, \alpha)\right], \qquad (5.9)$$

where

$$\rho^{(p)}_{2,k}(0, \alpha) = (\exp\{-\tfrac{1}{2}\chi^2_{p,\alpha}\})[(\tfrac{1}{2}\chi^2_{p,\alpha})^{k/2}/\overline{|k/2 + 1}](1 - \chi^2_{p,\alpha}/(k+2)), \qquad k \geqslant 1. \qquad (5.10)$$

Thus, if $\chi^2_{p.\alpha} \leqslant p + 2$ then $\rho_{2,p}(\Delta^*, \alpha)$ is $\geqslant 0$ for every $\Delta^* \geqslant 0$, while, if $\chi^2_{p.\alpha} > p + 2$, then there exists a $\Delta^*(p, \alpha) > 0$, such that $\rho_{2,p}(\Delta^*, \alpha)$ is $\leqslant$ or >0 according as Δ^* is $\leqslant$ or $>\Delta^*(p, \alpha)$. In any case, $|\rho_{2,p}(\Delta^*, \alpha)| \leqslant 1$ for every $\Delta^* \geqslant 0$ and $\alpha \in (0, 1)$ and $\Delta^*\rho_{2,p}(\Delta^*, \alpha) \to 0$ as $\Delta^* \to \infty$. Hence, for Δ^* close to 0 (depending on $\bar{t}^2/Q^*$, p and α), $e(\boldsymbol{\theta}^*, \hat{\boldsymbol{\theta}} \mid \boldsymbol{\gamma})$ is $\leqslant 1$, it exceeds 1 otherwise and it tends to ∞ as $\Delta^* \to \infty$. Thus, except when Δ^* is small, $\boldsymbol{\theta}^*_n$ is asymptotically relatively more efficient than $\hat{\boldsymbol{\theta}}_n$. In a similar manner, we have

$$\begin{aligned} e(\boldsymbol{\theta}^*, \tilde{\boldsymbol{\theta}} \mid \boldsymbol{\gamma}) &= [(1 + \bar{t}^2/Q^*)/(1 + (\bar{t}^2/Q^*)\{\rho_{1,p}(\Delta^*, \alpha) \\ &\qquad - \rho_{2,p}(\Delta^*, \alpha)\,\Delta^*\})(1 + \rho_{1,p}(\Delta^*, \alpha)\,\bar{t}^2/Q^*)^{p-1}]^{1/p}. \qquad (5.11) \end{aligned}$$

At $\Delta^* = 0$, (5.11) is >1, $\bar{t} \neq 0$ and $0 < \alpha < 1$. Also, as $\Delta^* \to \infty$, it converges to 1. For $p = 1$, (5.11) has already been studied in detail by Saleh and Sen [4] [viz. their (5.12)–(5.14)]. Hence, we confine ourselves here to $p \geqslant 2$. From our discussions following (5.8), we conclude that for Δ^* close to 0, $\rho_{1,p}(\Delta^*, \alpha) - \Delta^*\rho_{2,p}(\Delta^*, \alpha)$ is <1 and hence, $\boldsymbol{\theta}^*_n$ is asymptotically relatively more efficient and, also for large Δ^*, they are asymptotically equally efficient.

In passing, we may remark that for $\bar{t} \neq 0$, $\boldsymbol{\xi}_2(\boldsymbol{\gamma}) = \mathbf{0}$ but $\boldsymbol{\xi}_1(\boldsymbol{\gamma}) = \bar{t}\boldsymbol{\gamma}$ and $\boldsymbol{\xi}^*(\boldsymbol{\gamma}) = \{\bar{t}H_{p+2}(\chi^2_{p,\alpha}\,; \Delta^*)\}\boldsymbol{\gamma}$ where $H_{p+2}(\chi^2_{p,\alpha}\,; \Delta^*)$ is less than $1 - \alpha$ for all $\Delta^* \geqslant 0$ and it tends to 0 as $\Delta^* \to \infty$. Thus, $\hat{\boldsymbol{\theta}}_n$ has a comparatively larger asymptotic bias than $\boldsymbol{\theta}_n^*$ for all $\boldsymbol{\gamma}$, and, in particular, for large Δ^*, this bias makes $\hat{\boldsymbol{\theta}}_n$ comparatively inefficient. In this respect, the picture is very similar to the univariate case, studied in detail by Saleh and Sen [4]; coordinatewise extensions of their univariate results remain true for the multivariate case as well.

The two-sample location model is a special case of (1.1) when $t_1 = \cdots = t_{n_1} = 1$, $t_{n_1+1} = \cdots = t_n = 0$, $n = n_1 + n_2$ and $n_1 \geqslant 1$, $n_2 \geqslant 1$. In this case, $\boldsymbol{\beta}$ stands for the difference of location (vectors) of the two df's. Whenever $n_1/n \to \rho$: $0 < \rho < 1$, $\bar{t}_n \to \bar{t} = \rho$, $Q^* = \rho(1 - \rho)$ and $0 < \bar{t}^2/Q^* = \rho/(1 - \rho) < \infty$. Thus, the t_i satisfy the assumed regularity conditions. In particular, for the case of equal sample size, $\rho = \frac{1}{2}$, so that $\bar{t}^2/Q^* = 1$. In Table I, we present some numerical values of (5.8) and (5.11) for typical values of Δ^*, α and p. For the range of (Δ^*, α, p), considered in Table I, $\boldsymbol{\theta}_n^*$ appears to be asymptotically more efficient than $\tilde{\boldsymbol{\theta}}_n$. For small α and Δ^* close to 0, $\hat{\boldsymbol{\theta}}_n$ is asymptotically more efficient, while, the opposite conclusion holds when Δ^* increases. Similar pictures can be obtained for other values of ρ.

TABLE I

The A.R.E. in the Two-Sample (Equal Sample Size) Case

(α, Δ^*)		$e(\boldsymbol{\theta}^*, \hat{\boldsymbol{\theta}} \mid \boldsymbol{\gamma})$			$e(\boldsymbol{\theta}^*, \tilde{\boldsymbol{\theta}} \mid \boldsymbol{\gamma})$		
		$p = 2$	$p = 3$	$p = 4$	$p = 2$	$p = 3$	$p = 4$
$\alpha = .01$	$\Delta^* = 0$	0.947	0.957	0.962	1.892	1.914	1.924
	1	1.238	1.140	1.107	1.750	1.810	1.861
	2	1.408	1.230	1.170	1.626	1.706	1.777
	3	1.526	1.279	1.199	1.526	1.611	1.695
	4	1.619	1.305	1.232	1.448	1.527	1.648
$\alpha = .05$	$\Delta^* = 0$	0.833	0.858	0.871	1.667	1.716	1.742
	1	1.063	0.989	0.960	1.503	1.570	1.617
	2	1.213	1.054	0.993	1.400	1.462	1.510
	3	1.334	1.096	1.008	1.334	1.381	1.425
	4	1.441	1.129	1.015	1.290	1.320	1.357
$\alpha = .10$	$\Delta^* = 0$	0.750	0.780	0.797	1.500	1.560	1.594
	1	0.974	0.902	0.875	1.377	1.431	1.471
	2	1.131	0.969	0.908	1.306	1.344	1.380
	3	1.262	1.018	0.928	1.262	1.282	1.312
	4	1.377	1.056	0.941	1.232	1.235	1.259

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